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ON SMOOTH NORMS AND ANALYTIC SETS

BY

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ABSTRACT

Let X be a Banach space which isn't reflexive but has a separable dual. Then X admits a smooth norm so that the set of norm-attaining functionals is a complete analytic set. A variant of Asplund's average norms is used.

Introduction

Let X be a separable Banach space with norm $|\cdot|$; NA, or $NA(|\cdot|)$, is the set of linear functionals on X which attain their norm on the closed sphere of X; and NA_1 is the intersection of NA with the sphere of X^*. It is proved in [5] that if X isn't reflexive, then X admits an equivalent norm such that NA isn't a Borel set (for the metric topology of X^*) and in [2] this is accomplished with a Gâteaux-smooth norm. Neither of these offers any clues about spaces admitting a Fréchet-smooth (F-smooth) norm. We prove that NA fails to be a Borel set in a very definite way, with a smooth norm.

THEOREM: *Let X be a non-reflexive Banach space and admit an F-smooth norm. Then there is an equivalent F-smooth norm with this property: whenever N is a Polish space and $\mathcal{A}$ an analytic set in N, there is a continuous map φ of N into X^* such that $\mathcal{A} = \varphi^{-1}(NA_1)$.*

In spite of a formal similarity between Fréchet-smoothness and Gâteaux-smoothness, the former is much more difficult to attain in re-norming, and usually has to be approached through a dual norm on X^*. A separable space admits an F-smooth norm if and only if X^* is separable [Kadec; cf. 3, pp. 44–51], and then X admits a norm whose dual is LUR : $\|x^*\| = 1$, $\|x_n^*\| = 1$, $\lim \|x^* + x_n^*\| = 2$

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implies $\lim x_n = x$ [3, p. 42]. When the dual norm is LUR, then the norm is F-smooth [3, p. 43] and the main theorem relies on this observation.

Matters are much simpler in a special case: X contains a direct sum $Y_1 \oplus Y_2$ in which $\dim Y_1 = +\infty$ and Y_2 is non-reflexive. Since this isn't true in general (for example in spaces which are hereditarily indecomposable [4]), we use a substitute: there is a set M, homeomorphic to the Hilbert cube, such that $|m + y_2| \geq |m|$ for each $m \in M$ and $y_2 \in Y_2$ and the function $|m + y_2|$ has a strong minimum at $y_2 = 0$. This has to be combined with further properties of the norm, a process accomplished by the category method [3, p. 52] a variant of the "averaged norms" of Asplund [1].

In the final step towards finding the norm $\|\cdot\|$, and verifying that certain functionals do not belong to NA for this norm, we use the presence of a certain kind of closed, convex set found only in non-reflexive spaces. This follows [3], but derives from James' theorem on NA.

1. Let $f(u) = (1 - u^2)/4$, $0 \leq u \leq 1$. We use this function to form a sum of Banach spaces Z_1 and Z_2 with some properties of the ℓ^2-sum. We define

$$|(z_1, z_2)|' = \max u|z_1| + f(u)|z_2|, \quad 0 \leq u \leq 1.$$

A linear functional has norm at most 1 if $|z_1^*| \leq u$ and $|z_2^*| \leq f(u)$ for some u in $[0,1]$. More generally let K be the closed, convex set in R^2 defined by $0 \leq u \leq 1$, $0 \leq v \leq f(u)$: then (z_1^*, z_2^*) has norm at most 1 provided $(|z_1^*|, |z_2^*|) \in K$. Since K is convex, the separation theorem shows that $(|z_1^*|, |z_2^*|) \in K$ is necessary as well as sufficient. We observe next that when $|z_2^*| < f(|z_1^*|)$ then $|z_1^*| < 1$ and there is some $t > 1$ such that $t|z_2^*| < f(t|z_1^*|)$ so (z_1^*, z_2^*) has norm $\leq t^{-1} < 1$. Hence the unit sphere of the dual space is defined by the conditions $|z_1^*| \leq 1$, $|z_2^*| = f(|z_1^*|)$.

Clearly $|(z_1, z_2)'| \geq |z_1|$. When $2|z_1| \geq |z_2|$ then $u|z_1| + f(u)|z_2|$ is non-decreasing on $[0,1]$ and $|(z_1, z_2)|' = |z_1|$. Suppose now that z_1^* and z_2^* are LUR. From our identification of the unit ball of the dual space of $Z_1 \oplus Z_2$ with the norm $|\cdot|'$, we see that it too is LUR, since f is strictly concave on $[0,1]$. From this we see that the same is true of W^*, W being any subspace of $Z_1 \oplus Z_2$; we observe that the canonical quotient mapping of $(Z_1 \oplus Z_2)^*$ onto W^* maps the closed unit ball of the first space *onto* that of the second.

We apply this as follows. Let Y be a closed subspace of X, of infinite codimension, and $|\cdot|$ a norm on X, whose dual is LUR. We set $Z_1 = X/Y$ with the quotient norm (written with the same notation) so that Z_1^* is LUR. Let π be the projection of X on X/Y, and let W be the closed subspace of $X/Y \oplus X$ consisting of all elements $(\pi(x), x)$. Clearly W is isomorphic to X, with norm equal to $|x|_1 \equiv \sup u|\pi(x)| + f(u) \cdot |x|$. Besides the two inequalities on this norm written above, we observe that $|\pi(x)|_1 = |\pi(x)|$. (These are quotient norms on X/Y.) In fact, from $|x|_1 \geq |\pi(x)|$ we find $|\pi(x)|_1 \geq |\pi(x)|$. When $\pi(x) \neq 0$ choose $y \in Y$ so that $|x+y| < 2|\pi(x)|$; then $|x+y|_1 = |\pi(x+y)| = |\pi(x)|$, proving our assertion.

We now add the requirement that the quotient norm of $|\cdot|$ in X/Y be LUR to our previous requirement on X^*. This can be done by the category method [3, p. 52], which we shall illustrate later in a more complicated situation. To use the category method we need to know that X/Y admits at least one LUR norm [3, p. 42]. Of course the norm induced by $|\cdot|_1$ is LUR in X/Y since $|\pi(x)|_1 \equiv |\pi(x)|$.

2. We shall now construct a set M, homeomorphic to the Hilbert cube, such that π is a homeomorphism on M into X/Y, and $|m|_1 = |\pi(m)|_1$ for every element m of M. Let $(u_k)_1^\infty$ be a normalized basic sequence in X/Y, and let $u_k = \pi(x_k)$; this is possible because X/Y has infinite dimension. Since $|\pi(x_k)| = |\pi(x_k)|_1 = 1$ for each k, we can adjust x_k so that $|x_k| < 4/3$.

Let $\tilde{M}$ be the set of all sums $x_1 + \Sigma_2^\infty c_k x_k$, with $|c_k| \leq 3^{-k}$. We note that distinct elements of $\tilde{M}$ aren't proportional *modulo* Y, because all begin with x_1. The elements of $\tilde{M}$ satisfy $|\pi(\tilde{m})| \geq |\pi(\tilde{m})|_1 \geq 1 - 1/6$ and $|\tilde{m}| \leq 7/6 \cdot 4/3 < 5/3$. Since $|\tilde{m}| < 2|\pi(\tilde{m})|$ then $|\pi(\tilde{m})|_1 = |\tilde{m}|_1$; we obtain M by mapping an element $\tilde{m}$ to $|\tilde{m}|_1^{-1} \cdot \tilde{m}$; so that M and $\tilde{M}$ are homeomorphic. We observe that $\pi(M)$ doesn't meet its negative $-\pi(M)$.

When $m \in M$ and $y \in Y$ then $|m + ty|_1 = 1$ for small t. This would interfere with a later step in our argument, so it must be removed by an intermediate re-norming. This entails an application of the category method, and is the most intricate part of the theorem. We denote by J the duality mapping on X, using the norm $|\cdot|_1$. Since the norm in X^* is LUR, the mapping J is norm-to-norm continuous, except at 0 in X. When $x \in M$, then $J(x) \in Y^\perp$. We now see that when m_1 and m_2 belong to M and $m_1 \neq m_2$, then $|\langle J(m_1), m_2\rangle| < 1$. For $\pi(m_1) \neq \pm\pi(m_2)$, the norm in X/Y is rotund.

LEMMA (a): *For each ϵ in $(0,1)$, there is a number $r > 0$ with this property. Whenever $m \in M$, $y \in Y$, and $|y_1| = \epsilon$, then there is a linear functional f, of norm < 2, such that $|f| \leq 1$ on M, and $f(m+y) > 1 + r$.*

Proof: Since $\langle J(m), y\rangle = 0$ the distance from y to Rm is at least $\epsilon/2$. Hence there is a linear functional g, such that $g(y) \geq \epsilon/2$, $g(m) = 0$, and $|g|_1 \leq 1$. Let $\sigma > 0$ be so small that $|\langle J(x), m\rangle| < 1 - \sigma$ whenever $x, m \in M$ and $|x - m|_1 \geq \epsilon/4$. (Here we use the compactness of M, continuity of J, and the observation preceding the

Lemma.) We remark that σ depends only on ϵ. We set $f = (1-\sigma\epsilon/4)J(m)+\sigma g$ so that $f(m+y) \geq 1-\sigma\epsilon/4+\sigma\epsilon/2 = 1+\sigma\epsilon/4$. When $x \in M$ and $|f(x)| > 1$ then $|\langle J(m), x\rangle| > 1-\sigma$, whence $|x-m|_1 < \epsilon/4$. Therefore $|f(x)| < 1-\sigma\epsilon/4+\sigma\epsilon/4 = 1$, a contradiction. Moreover $|f|_1 < 1+\sigma < 2$; thus we can choose $r = \sigma\epsilon/4$. ∎

LEMMA (b): *There exists a norm $|\cdot|_2$ on X such that:*

(i) $|x|_1 \leq |x|_2 \leq 2|x|_1$ *for all x in X.*

(ii) $|m|_2 = 1$ *for each m in M.*

(iii) *The dual norm is LUR.*

(iv) *For each ϵ in $(0,1)$ there is some $\theta(\epsilon) > 0$ so that $|m+y|_2 \geq 1+\theta(\epsilon)$ whenever $m \in M$, $y \in Y$, $|y|_1 = \epsilon$. Here θ is allowed to depend on ϵ and the norm $|\cdot|_2$.*

Proof: In this lemma it will be convenient to denote norms by p, and $|\cdot|_1$ by p_1. We denote by A the set of norms p such that $p_1 \leq p \leq 2p_1$ and $p = 1$ on M. A is a complete metric space with the metric ρ defined by $\rho(p,q) = \sup\{|p(x)-q(x)|: |x|_1 \leq 1\}$. In the set A there is a largest norm, defined by the unit ball of its dual space: $q(x^*) \leq 1$ if and only if $|x^*| \leq 1$ on M and $|x^*| \leq 2$ on $B(X, |\cdot|_1)$. We define sequences $(U_n)_1^\infty$ and $(V_n)_1^\infty$ of dense open sets in A, and apply Baire's Theorem.

We say that $p \in U_n$ if there is a number $a_n > 0$ such that $p(m+y) \geq 1+a_n$ whenever $m \in M$, $y \in Y$, and $|y|_1 = 1/n$. Clearly U_n is open because the elements $m+y$ in question have norm at most 2. Lemma (a) shows that the extremal norm q belongs to each U_n because $|f| \leq q$ is valid for every functional f obtained there. Now $\lim N^{-1}p+(1-N^{-1})p = p$ and $N^{-1}p+(1-N^{-1})p \in U_n$ whenever $N \geq 1$, $n \geq 1$ (and each convex combination belongs to A). The set $\bigcap_1^\infty U_n = H$, say, is a dense G_δ in A, and all the norms in H have property (iv).

The definition of V_n is more complicated. We say $p \in V_n$ if there is some $N > n$, and norms q_1, q_2 in A such that $\rho(p, q_1) < N^{-2}$ and the dual norms satisfy the identities $q_1(x^*)^2 \equiv N^{-1}p_1(x^*)^2 + (1-N^{-1})q_2(x^*)^2$. Clearly V_n is open, and $G = \bigcap_1^\infty V_n$ contains only norms whose duals are LUR because p_1 has that property [3, p. 53]. When $p \in A$ and q_N is defined by $q_N(x^*)^2 = N^{-1}p_1(x^*)^2 + (1-N^{-1})p(x^*)^2$, we find that $\lim q_N = p$ and q_N can take the place of q_1. However, we have to verify that each $q_N \in A$; the inequality $p_1 \leq q_N \leq 2p_1$ is equivalent with $p_1^*/2 \leq q_N^* \leq p_1^*$, and this is a consequence of the corresponding inequalities on p_1 and p. To verify that $q_N \leq 1$ on M, we show that in general $q_N(x) \leq \max(p_1(x), p(x))$. From Cauchy's inequality we conclude $q_N(x^*) \geq N^{-1}p_1(x^*)+(1-N^{-1})p(x^*)$. The norm on the right is abbreviated by $\tilde{p}$. Then for all x in X^* we have $|x^*(x)| \leq \tilde{p}(x^*)\max(p_1(x), p(x))$, and therefore

$q_N(x) \leq \max(p_1(x), p(x))$. Thus the sequence q_N, which converges to p, remains in A, and each set V_n is dense in A. Any norm in $G \cap H$ serves for Lemma (b).

We can now prove that the elements of M are strongly exposed by the duality (or tangent) functionals, in the new larger norm $|\cdot|_2$. Since $|m|_1 = |m|_2 = 1$ for each m, and each norm is F-smooth, while $|\cdot|_1 \leq |\cdot|_2$, the functional $J(m)$ has the same meaning for both norms. In particular $J(m)$ has norm 1 for the smaller norm. Suppose now that $|z_n|_2 \leq 1$ and $\langle J(m), z_n\rangle \to 1$. Since $J(m) \in Y^\perp$ we see that $|\pi(m) + \pi(z_n)|_1 \to 2$ and so by the properties of the norm $|\cdot|_1$ in X/Y, $\lim \pi(z_n) = \pi(m)$ (and $|\pi(z_n)|_1 \leq 1$). Thus $z_n = m + y_n + w_n$, where $y_n \in Y$ and $\lim w_n = 0$, whence $|m + y_n|_2 \leq 1 + |w_n|_2 = 1 + o(1)$ and finally $\lim y_n = 0$ (by the Lemma) or $\lim z_n = m$. ∎

3. In the case that X isn't reflexive, we can choose Y of infinite codimension, and also nonreflexive, as follows. By a theorem of Pelczyński [6], X contains a bounded basic sequence (u_k) such that $f^*(v_k) = 1$, f^* being a certain bounded linear functional. The bounded sequence (v_k) has no weak accumulation point. For if w were an accumulation point, it would be in the null-space of the biorthogonal functionals for the basis; thus $w = 0$, contradicting $f^*(w) = 1$. Let $u_k = v_{2k}$ and $Y = sp(u_k)$. Then X/Y has infinite dimension, since Y is in the null space of infinitely many of the biorthogonal functionals, and these are linearly independent elements of X^*. Using the basic sequence (u_k), we let P_k be the usual projections of Y and $f_k^* = f^* - P_k^* f^*$. Then the f_k^* are uniformly bounded and we have

$$f_k^*(u_j) = 0, \quad 1 \leq j \leq k,$$
$$f_k^*(u_j) = 1, \quad 1 \leq k < j.$$

We introduce now the Baire null-space Σ, consisting of strictly increasing sequences $\sigma = (n_k)_1^\infty$ of natural numbers. We define $h(\sigma) = \Sigma_1^\infty 2^{-k} u_{n_k}$, and recall the following property of h [5]. Whenever $S = (\lambda_j)_1^\infty$ is a sequence of probability measures in Σ, and the integrals $\int h d\lambda_j$ belong to a compact set in X, then the sequence S is uniformly tight: for each $\epsilon > 0$ there is a compact set $K = K(\epsilon)$ such that $\lambda_j(K) > 1 - \epsilon$ for all j. This is seen by applying the functionals f_k^* to the sequence $\int h d\lambda_j$, and expressing the value as an integral $\int f_k^*(h(\sigma)) d\lambda_j$. We use the tightness to conclude that the closed convex hull of $h(\Sigma)$ consists of integrals $\int h d\lambda$, where λ is a probability measure in Σ.

Let u be a continuous map of Σ onto a dense subset of M, and $g = u + h$. (More details on u are presented below.) Now g has the same property as h, concerning tightness, because the addition of h to u doesn't affect the compactness.

The norm $\|\cdot\|$ is defined through its dual norm. We define $p(x^*)$ to be the supremum of $|x^*|$ over $g(\Sigma)$ and then $\|x^*\| = p(x^*) + |x^*|_2$, so that $\|\cdot\|$ is indeed a dual norm. To see that it is LUR, suppose that $\|x^*\| = 1$, $\|x_n^*\| = 1$, and $\lim \|x^* + x_n^*\| = 2$. Since p is subadditive it follows that

$$\lim |x_n^* + x^*|_2 - |x_n^*|_2 - |x^*|_2 = 0.$$

Let t_n be defined by $|x_n^*|_2 = t_n|x^*|_2$. Then $\lim |t_n x_n^* + x^*|_2 - t_n|x_n^*|_2 - |x^*|_2 = 0$ and so $\lim t_n x_n^* = x^*$. Since $\|x_n^*\| = \|x^*\| = 1$, $\lim t_n = 1$ and $\lim x_n^* = x^*$.

We study the norm-attaining properties of the functionals $J(m)$, $m \in M$; here $J(m)$ refers to the norm $|\cdot|_2$, but norm-attaining refers to the norm $\|\cdot\|$ defined above. Now $h(\Sigma) \subseteq Y$, and so $J(m)$ vanishes on $h(\Sigma)$, while $u(\Sigma)$ is dense in M. Thus $p(J(m)) = 1$ and $\|J(m)\| = 2$. Moreover each element $g(\sigma) + u(\sigma) = 2u(\sigma) + h(\sigma)$ has $\|\cdot\|$-norm at most 1, since $|x^*(g(\sigma))| \leq p(x^*)$ and $|x^*(u(\sigma))| \leq |x^*|_2$. The value of $J(m)$ on this element is $2\langle J(m), u(\sigma)\rangle$ (because $h(\Sigma) \subseteq Y$); thus $J(m)$ attains its norm if $m = u(\sigma)$ for some σ, i.e. $m \in u(\Sigma)$. We shall now embark on proving the converse implication, making critical use of the special property of h.

First of all, we can find the unit ball of $\|\cdot\|$ from the bipolar theorem. It consists of the closure of sums $w + z$ where $w \in co(\pm g(\Sigma))$ and $|z|_2 \leq 1$. Suppose $w_n + z_n$ is a convergent sequence of such sums and that $J(m)$ tends to 2 on this sequence. Then $\langle J(m), z_n\rangle \to 1$ so that $z_n \to m$. Therefore $w = \lim w_n$ exists and $\langle J(m), w\rangle = 1$. The value of $J(m)$ at $g(\sigma)$ is $\langle J(m), u(\sigma)\rangle$ and this is at least 1/2 because M has diameter $< 1/2$. Each w_n is (formally) an integral $\int g(\sigma) d\lambda_n$, where λ_n is a signed measure of variation at most 1. Our observation on $\langle J(m), g(\sigma)\rangle$ implies that the negative variation of λ_n must tend to 0, i.e. $\lambda_n(\Sigma) \to 1$. Thus we can replace λ_n by a probability measure in the following. Since $\lim w_n$ exists we see that the sequence (λ_n) is uniformly tight, and has a limit λ, concentrated in Σ. But then $w = \int g(\sigma) d\lambda$, and $\langle J(m), g(\sigma)\rangle$ must attain the value 1 on Σ, i.e. $m \in u(\Sigma)$. Thus we have found that $J(m)$ is in NA for the norm $\|\cdot\|$ if and only if $m \in u(\Sigma)$.

4. The Hilbert cube Q contains a nowhere dense, compact subset Q_1, homeomorphic to Q itself. Let Q_2 be a subset of $C[0,1]$ defined as follows: $v \in Q_2$ if $0 \leq v \leq 1$ and $|v(s) - v(t)| \leq |s - t|$, $0 \leq s < t \leq 1$. Let $\mathcal{A}'$ be the analytic subset of Q_2, consisting of functions having an irrational zero. We treat Q_2 as a closed subset of Q_1; this is possible for any compact metric space. Then $\mathcal{B} = \mathcal{A}' \cup (Q \backslash Q_2)$ and so $\mathcal{B} = u(\Sigma)$ with a continuous mapping u of Σ; clearly $\mathcal{B}$ is dense in Q. We accept for a moment that $\mathcal{A}'$ reduces any analytic set $\mathcal{A}$ by a continuous map ψ into Q_2. The mapping φ defined by $\varphi(t) \equiv 1/2 \cdot J(\psi(t))$ then

has the property that $\varphi^{-1}(NA) = \varphi^{-1}(NA_1) = \mathcal{A}$. In this assertion NA and NA_1 refer to the norm $\|\cdot\|$, while J refers to the norm $|\cdot|_2$.

To explain the point left open about $\mathcal{A}'$, let N be a Polish space of diameter < 1, let $\mathcal{A}$ be analytic in N, and $\mathcal{A} = \theta(I)$, I the set of irrationals in $(0,1)$. We then define $v(x,t) = 1/2 \inf\{d(x,\theta(s)) + |t-s| : s \in I\}$. Then the partial function $v(\cdot,t)$ belongs to Q_2 for every x in N, and $v(\cdot,t)$ belongs to $\mathcal{A}'$ — that is, vanishes at an irrational — if and only if $x \in \mathcal{A}$. Moreover, $v(x,t)$ is Lipschitz-continuous in the variable x. This is the map ψ on N into Q_2.

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